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LATTICE POINTS AND LIE GROUPS. II

BY

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ABSTRACT. Let C be the Casimir operator on a compact, simple, simply connected Lie group G of dimension n. The number of eigenvalues of C, counted with their multiplicities, of absolute value less than or equal to t is asymptotic to $kt^{n/2}$, k a constant. This paper shows the error of this estimate to be $O(t^{2b+a(a-1)/(a+1)})$; where $a =$ rank of G and $b = \frac{1}{2}(n-a)$.

Let G be a compact, simple, simply connected Lie group and $\mathfrak{G}$ its Lie algebra. Following the terminology of *Lattice points and Lie groups*. I we have $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{G}$, $\mathfrak{H}^*$ its dual, (,) the Killing form, R the roots of $\mathfrak{G}$, $a_{\mathfrak{G}} = \dim \mathfrak{H}$, $b_{\mathfrak{G}} = \frac{1}{2}|R|$.

The Lie algebra of a Lie group is defined as the left or right invariant tangent vector fields. If we choose to consider them as left invariant then $U(\mathfrak{G})$, the universal enveloping algebra, becomes identified with the left invariant differential operators and $U(\mathfrak{G})$ acts on $C^\infty(G)$.

If $\mathfrak{G}$ is a simple Lie algebra, the Casimir operator C is defined as $\sum_{i,j=1}^{a} g^{ij}X_iX_j$ where the X_i form a base for and g^{ij} are the elements of the matrix $(g_{ij})^{-1}$ where $g_{ij} = (X_i, X_j)$. This definition is independent of the basis chosen, and since G is compact, the matrix (g_{ij}) represents a nondegenerate negative definite quadratic form. Thus (g^{ij}) represents a negative definite form and the operator C is elliptic. The asymptotic behavior of the eigenvalues of elliptic operators have been studied in great generality. If D is a selfadjoint elliptic operator defined on a compact manifold and $E(D, T)$ equals the number of eigenvalues of D less than or equal to T counted with their multiplicities then $E(D, T) \sim kT^{n/m}$ where n is the dimension of the manifold and m is the degree of D [1]. For open bounded subsets of R^n and D with smooth coefficients the error for this estimate is $O(T^{n/m-\frac{1}{2}m+\epsilon})$ for any $\epsilon > 0$ [2]. For compact subsets of manifolds with D again smooth, the error is $O(T^{(n-1)/m})$ [4]. However, for C the Casimir operator of a compact, simple, simply connected Lie group a better estimate may be made.

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The left-regular representation of G is a direct sum of all finite-dimensional irreducible representations of G, each occurring with multiplicity equal to its dimension, and the functions in the representation spaces may be assumed to be in $C^{\infty}(G)$. Also, C, regarded as a differential operator, commutes with each representation, so each representation space is an eigenspace of C. Moreover, as we shall see, the corresponding eigenvalue depends only on the representation, and can be explicitly calculated in terms of the corresponding dominant weight λ. Finally, if G is simply connected, there is a bijection between representations of G and $\mathfrak{G}$. Thus, using the terminology of [1], $E(C, T)$ or for brevity $E(T)$, equals

$$\sum_{\Lambda \geq \delta, c_\Lambda \leq T} f_{\mathfrak{G}}^2(\Lambda)$$

where $f_{\mathfrak{G}}(\Lambda)$ is the Weyl dimension polynomial and $\pi_\lambda(C) = c_\Lambda \mathrm{Id}$, $\Lambda = \lambda + \delta$. Since $f_{\mathfrak{G}} = 0$ along the walls of the positive Weyl chamber P, this sum equals

$$\sum_{\Lambda \geq 0, c_\Lambda \leq T} f_{\mathfrak{G}}^2(\Lambda).$$

To clarify this sum we wish to compute c_Λ. To do this fix a Cartan subalgebra $\mathfrak{H} \subset \mathfrak{G}$ and $\alpha_1, \cdots, \alpha_a$ simple roots. Each $\mathfrak{G}_\alpha$ is one-dimensional so pick $X_\alpha \in \mathfrak{G}_\alpha$ for all roots α and normalize the X_α so that $(X_\alpha, X_{-\alpha}) = 1$. Then $H_\alpha = [X_\alpha, X_{-\alpha}]$. Pick the basis for $X_i = H_{\alpha_i}$, $i = 1, \cdots, a$, and $X_i = X_\alpha$, $i = a+1, \cdots, n$. Then $g_{ij} \neq 0 \Rightarrow i, j \leq a$ or that $X_i = X_\alpha$ and $X_j = X_{-\alpha}$, and in this case $g_{ij} = 1$. From this it is seen

$$C = \sum_{i,j=1}^{a} g^{ij} H_{\alpha_i} H_{\alpha_j} + \sum_{\alpha} X_\alpha X_{-\alpha} = \sum_{i,j=1}^{a} g^{ij} H_{\alpha_i} H_{\alpha_j} + \sum_{\alpha > 0} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)$$

$$= \sum_{i,j=1}^{a} g^{ij} H_{\alpha_i} H_{\alpha_j} + \sum_{\alpha > 0} 2X_{-\alpha} X_\alpha + \sum_{\alpha > 0} H_\alpha.$$

Pick $v \in V_\lambda$. Since λ is the dominant weight of π_λ, $\pi_\lambda(X_\alpha)v = 0$, $\alpha > 0$, so

$$\pi_\lambda(C)v = \left\{ \sum_{i,j=1}^{a} g^{ij} \lambda(H_{\alpha_i}) \lambda(H_{\alpha_j}) \right\} v + \sum_{\alpha > 0} \lambda(H_\alpha) v = \left(|\lambda|^2 + \sum_{\alpha > 0} \lambda(H_\alpha) \right) v$$

$$= \left(|\lambda|^2 + 2 \cdot \frac{1}{2} \sum_{\alpha > 0} (\lambda, \alpha) + |\delta|^2 - |\delta|^2 \right) v = (|\Lambda|^2 - |\delta|^2) v.$$

Thus $c_\Lambda = |\Lambda|^2 - |\delta|^2$ and

$$E(T) = \sum_{\Lambda \geq 0, |\Lambda| \leq \mathfrak{T}} f_{\mathfrak{G}}^2(\Lambda) \quad \text{where } \mathfrak{T} = (T + |\delta|^2)^{1/2}.$$

The sum runs over the lattice points in a solid angle of the a-ball in $B(\mathfrak{T})$ in $\mathfrak{H}^*$.

We would like, for analytical purposes, to have the sum extend over the whole of $B(\mathfrak{T})$ so we build up $f_{\mathfrak{G}}^2$ by the action of the Weyl group. The Weyl group W and its properties are found in [8, Chapter 5]. It is generated by reflections so that W maps $B(\mathfrak{T})$ into itself and W is simply transitive on the Weyl chambers of $\mathfrak{H}^*$ [5, p. 242]. From this it follows that $f^2_{\mathfrak{G}}$ is invariant under W. Let μ and μ' be two sets of simple roots. Then there exists a unique $w \in W$ such that $w\mu = \mu'$. Let

$$f_{\mathfrak{G},\mu}(\Lambda) = c \prod_{\alpha>0 \text{ relative to } \mu} (\Lambda, \alpha)$$

where $\Lambda > 0$ relative to μ.

$$f_{\mathfrak{G},\mu'}(\Lambda) = c \prod_{\alpha>0 \text{ relative to } \mu'} (\Lambda, \alpha)$$

where $\Lambda > 0$ relative to μ'.

$$f^2_{\mathfrak{G},\mu}(\Lambda) = \left(c \prod_{\alpha>0 \text{ relative to } \mu} (\Lambda, \alpha)\right)^2$$

$$= \left(c(-1)^{\text{length of } w} \prod_{\alpha>0 \text{ relative to } \mu} (w\Lambda, w\alpha)\right)^2$$

$$= \left(c \prod_{\alpha>0 \text{ relative to } \mu'} (w\Lambda, \alpha)\right)^2 = f^2_{\mathfrak{G},\mu'}(w\Lambda).$$

Thus f^2 may be regarded as a single function on $B(\mathfrak{T})$ and

$$\sum_{\Lambda>0, \Lambda\in B(\mathfrak{T})} f^2_{\mathfrak{G}}(\Lambda) = \frac{1}{|W|} \sum_{\Lambda\in B(\mathfrak{T})} f^2_{\mathfrak{G}}(\Lambda).$$

Normalizing Lebesgue measure such that the volume of the fundamental domain of Λ is 1, the sum $\Sigma_{\Lambda\in B(\mathfrak{T})} f^2_{\mathfrak{G}}(\Lambda)$ will be asymptotic to $\int_{B(\mathfrak{T})} f^2_{\mathfrak{G}}(x)\,dx$ since $f^2_{\mathfrak{G}}$ is homogeneous. Taking the coordinate change $y = x/\mathfrak{T}$ then $dy = dx/\mathfrak{T}^a$ and this integral equals

$$\int_{B(1)} f^2(\mathfrak{T}y)\mathfrak{T}^a\,dy$$

and, since f is homogeneous of degree b,

$$= \int_{B(1)} f^2_{\mathfrak{G}}(y)\,\mathfrak{T}^{2b}\mathfrak{T}^a\,dy$$

and, since $\dim G = 2b + a$,

$$= \mathfrak{T}^{\dim G} \int_{B(1)} f^2_{\mathfrak{G}}(y)\,dy.$$

The latter integral is a constant which we will call $d_{\mathfrak{G}}$. Therefore

$$E(T) = d_{\mathfrak{G}}/|W|\,\mathfrak{T}^{\dim G} + o(\mathfrak{T}^{\dim G}).$$

The purpose of this paper will be to estimate $R(T) = \big|\,|W|E(T) - d_{\mathfrak{G}}\mathfrak{T}^{\dim G}\big|$.

The most elementary way of estimating $R(T)$ is to note that $R(T)$ is less than $f_{\mathfrak{G}}^2(y)$ integrated over a shell of width $2a^{1/2}$ around the boundary of $B(\mathfrak{T})$. This, in turn, is less than the maximum of $f_{\mathfrak{G}}^2(y)$ in this region, $k\mathfrak{T}^{2b}$, times the volume, which is $k'\mathfrak{T}^{a-1}$. Therefore $R(T)$ is $O(\mathfrak{T}^{\dim G-1})$. However, for better results we will have to turn to analytic techniques using the Poisson summation formula. The methods are adaptations of those of Randol and his work will be referred to in the following pages.

Theorem. *If G is a compact, simple, simply connected Lie group then $R(T)$ is $O(\mathfrak{T}^{2b+a(a-1)/a+1})$.*

Proof. Let $g_T(x) = f_{\mathfrak{G}}^2(x)\chi_{B(\mathfrak{T})}(x)$ where $\chi_{B(\mathfrak{T})}(x)$ is the characteristic function of the ball of radius $\mathfrak{T}$. Since $f_{\mathfrak{G}}^2(x)$ is analytic $g_T(x)$ would be of rapid decrease were there not a discontinuity on the boundary of $B(\mathfrak{T})$. To remove this we convolve $g_T(x)$ with a smoothing function. Let $\delta_1(x)\colon \mathbf{R}^a \to \mathbf{R}^+$ with support on $B(1)$ and $\int_{\mathbf{R}^a}\delta_1(x)\,dx = 1$. Define $\delta_\epsilon(x) = (1/\epsilon^a)\delta_1(x/\epsilon)$. Then δ_ϵ has support on $B(1)$ and $\int_{\mathbf{R}^a}\delta_\epsilon(x)\,dx = 1$. Define

$$h_T(x) = g_T * \delta_\epsilon(x).$$

Then h_T is C^∞ and of compact support. If L is the lattice $\mathbf{Z}\lambda_1 \oplus \cdots \oplus \mathbf{Z}\lambda_n$ and $\hat{L}$ is the dual lattice in $\hat{\mathbf{R}}^a = \mathbf{R}^a$ we have

$$\sum_{n\in L} h_T(n) = \sum_{n\in\hat{L}} \hat{h}_T(n) = \sum_{n\in\hat{L}} \hat{\delta}_\epsilon(n)\hat{g}_T(n) = \hat{\delta}_\epsilon(0)\hat{g}_T(0) + \sum_{n\in\hat{L}}{}' \hat{\delta}_\epsilon(n)\hat{g}_T(n)$$

where Σ' means summation over all lattice points except the origin. Now $\hat{\delta}_\epsilon(0) = \int_{\mathbf{R}^a}\delta_\epsilon(x)\,dx = 1$ and

$$\hat{g}_T(0) = \int_{\mathbf{R}^a} f_{\mathfrak{G}}^2(x)\chi_{B(\mathfrak{T})}(x)\,dx = \int_{B(\mathfrak{T})} f_{\mathfrak{G}}^2(x)\,dx = d_{\mathfrak{G}}\mathfrak{T}^{\dim G}.$$

Thus $\Sigma_{n\in L}\, h_T(n) = d_{\mathfrak{G}}\mathfrak{T}^{\dim G} + \Sigma'\,\hat{\delta}_\epsilon(n)\hat{g}_T(n)$. By the triangle inequality

$$R(T) \le \left|\sum_{n\in L} h_T(n) - d_{\mathfrak{G}}\mathfrak{T}^{\dim G}\right| + \left||W|\,E(T) - \sum_{n\in L} h_T(n)\right|$$

(1)

$$= \sum_{n\in\hat{L}} |\hat{\delta}_\epsilon(n)\hat{g}_T(n)| + \sum_{n\in L} |h_T(n) - g_T(n)|.$$

We wish to prove both sums $O(\mathfrak{T}^{2b+a(a-1)/a+1})$.

We first estimate the second sum. If $|n| > \mathfrak{T} + \epsilon$ then $g_T(n) = h_T(n) = 0$. If $|n| < \mathfrak{T} - \epsilon$ then

$$|h_T(n) - g_T(n)| \le \max_{y\in B(\epsilon)} |g_T(n+y) - g_T(n)| = |\mathrm{Grad}\, f_{\mathfrak{G}}^2(x')|\epsilon$$

for some x' such that $d(x', n) \le \epsilon$. Since $f_{\mathfrak{G}}(x)$ is homogeneous of degree b, $|\mathrm{Grad}\, f^2(x')| \le k\mathfrak{T}^{2b-1}$, so that $|h_T(n) - g_T(n)| \le k\mathfrak{T}^{2b-1}\epsilon$. There are less than $k'\mathfrak{T}^a$ lattice points in $B(\mathfrak{T}-\epsilon)$ so $\Sigma_{n\in B(\mathfrak{T}-\epsilon)}|h_T(n) - g_T(n)|$ is $O(\mathfrak{T}^{2b+a-1}\epsilon)$.

If $\mathfrak{T} + \epsilon \geq |n| \geq \mathfrak{T} - \epsilon$ then $|h_T(n) - g_T(n)| \leq k\mathfrak{T}^{2b}$. By [3] the number of lattice points in $B(\mathfrak{T}) = c\mathfrak{T}^a + O(\mathfrak{T}^{a(a-1)/a+1})$. Thus the number of lattice points in $B(\mathfrak{T} + \epsilon) - B(\mathfrak{T} - \epsilon)$ is $O(\mathfrak{T}^{a-1}\epsilon)$ if $\epsilon \geq \mathfrak{T}^{-(a-1)/(a+1)}$. With this restriction on ϵ, $\sum_{n \in B(\mathfrak{T}+\epsilon) - B(\mathfrak{T}-\epsilon)} |h_T(n) - g_T(n)|$ is $O(\mathfrak{T}^{2b+a-1}\epsilon)$. This leaves

$$\sum_{n \in \hat{L}}{}' |\hat{\delta}_\epsilon(n)\hat{g}_T(n)| \tag{2}$$

to analyze.

We begin by noting

$$\hat{\delta}_\epsilon(x) = \epsilon^{-a}\widehat{\delta_1(x/\epsilon)} = \hat{\delta}_1(\epsilon x).$$

Since δ_1 is of rapid decrease in x, $\hat{\delta}_\epsilon$ is of rapid decrease in ϵx. The tricky point is the behavior of $\hat{g}_T(x)$. By the Riemann-Lebesgue lemma $\hat{g}_T(x) \to 0$ as $|x| \to \infty$ but to estimate (2) we need much better estimates for $\hat{g}_T(x)$.

Lemma. *Let $f \in C^\infty(B^a(p))$. Then $\widehat{f \cdot \chi_{B(p)}}(z)$ is $O(|z|^{-(a+1)/2})$.*

Proof. If $f = 1$ this is well known [3]. By definition

$$\widehat{f \cdot \chi_{B(p)}}(z) = \int_{B(p)} e^{2\pi i(x,z)} f(x)\,dx.$$

We would like to transform this into a boundary integral by use of the divergence theorem. We wish to show that there exists a vector field $(g_i(x, z))$ such that

$$f(x)e^{2\pi i(x,z)} = \operatorname{div}_x((e^{2\pi i(x,z)}/2\pi i|z|) \cdot (g_i(x, z))) \tag{3}$$

where the $g_i(x, z)$ and their first m derivatives can be bounded in terms of f and its first m derivatives independent of z. Expanding the right side of (3):

$$\operatorname{div}_x((e^{2\pi i(x,z)}/2\pi i|z|)(g_i(x, z)))$$
$$= (e^{2\pi i(x,z)}/2\pi i|z|)\left[\sum_{i=1}^{a} (2\pi i z_i g_i(x, z)) + \partial g_i/\partial x_i(x, z)\right].$$

The latter expression equals $f(x)e^{2\pi i(x,z)}$ if the two equations

$$\sum_{i=1}^{a} \partial g_i/\partial x_i(x, z) = 0,$$

$$\sum_{i=1}^{a} \beta_i g_i(x, z) = f(x), \quad \text{where } \beta_i = z_i/|z|,$$

can be solved simultaneously. Such a solution of class C^∞ is found in [7, Lemma 3]. Thus

$$\int_{B(p)} f(x)e^{2\pi i(x,\,z)}\,dx = \frac{1}{2\pi i|z|}\int_{\partial B(p)} e^{2\pi i(y,z)} g_1(y,\,z)\,dy$$

where $g_1(y,\,z) = (g(y,\,z),\,n(y))$; $n(y)$ is the unit normal at y. The lemma now follows from the estimate for Fourier transforms on p. 766 of [6].

We wish to use the lemma to estimate (2).

$$\hat{g}_T(\Lambda) = \int_{B(\mathfrak{T})} e^{2\pi i(x,\Lambda)} f_{\mathfrak{G}}^2(x)\,dx.$$

Let $x = \mathfrak{T}y$ then $dx = \mathfrak{T}^a\,dy$ and

$$\hat{g}_T(\Lambda) = \int_{B(1)} e^{2\pi i(\mathfrak{T}y,\Lambda)} f_{\mathfrak{G}}^2(\mathfrak{T}y)\mathfrak{T}^a\,dy$$

$$= \int_{B(1)} e^{2\pi i(y,\mathfrak{T}\Lambda)} f_{\mathfrak{G}}^2(y)\mathfrak{T}^{2b}\mathfrak{T}^a\,dy$$

$$= \mathfrak{T}^{2b+a}\hat{g}_1(\mathfrak{T}\Lambda).$$

By the lemma $\hat{g}_1(\mathfrak{T}\Lambda) \le k|\mathfrak{T}\Lambda|^{-(a+1/2)}$ so

$$|\hat{g}_T(\Lambda)| \le k\mathfrak{T}^{2b+(a-1)/2}|\Lambda|^{-(a+1)/2}.$$

As was mentioned before, $\hat{\delta}_\epsilon(\Lambda)$ is of rapid decrease in $\epsilon\Lambda$ so there exists k_1 such that

$$\hat{\delta}_\epsilon(\Lambda) \le k_1/(1+|\epsilon\Lambda|)^a \quad \text{for any } \Lambda \ne 0.$$

Then

$$\sum_{\Lambda\in\hat{L}}{}' |\hat{\delta}_\epsilon(\Lambda)\hat{g}_T(\Lambda)| \le k_2\,\mathfrak{T}^{2b+(a-1)/2}\sum_{\Lambda\in\hat{L}}{}' |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a}.$$

We now choose $\epsilon = \mathfrak{T}^{-(a+1)/(a+1)}$. To compute $\sum'_{n\in\hat{L}}|\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a}$ we divide it into two sums

$$\sum_{|\Lambda|\le 1/\epsilon}{}' |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a} + \sum_{|\Lambda|>1/\epsilon} |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a}.$$

Comparing these sums to integrals

$$\sum_{|\Lambda|\le 1/\epsilon}{}' |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a} \le k\int_1^{1/\epsilon} x^{-(a+1)/2}\cdot x^{a-1}\,dx + k'$$

where k accounts for the volume of the parallelopiped spanned by the λ_i and k' for the sum of all Λ such that $|\Lambda| < 2$. Thus

$$\sum_{|\Lambda|\le 1/\epsilon}{}' |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a} \le k\int_1^{1/\epsilon} x^{(a-3)/2}\,dx + k'$$

$$= 2k/(a-1)x^{(a-1)/2}\Big|_1^{\mathfrak{T}^{(a-1)/(a+1)}} + \kappa' = 2k/(a-1)\mathfrak{T}^{(a-1)^2/2(a+1)} + k'.$$

Similarly

$$\sum_{|\Lambda|>1/\epsilon} |\Lambda|^{-(a+1)/2}(1+|\epsilon\Lambda|)^{-a} \le k\int_{1/\epsilon}^{\infty} x^{-(a+1)/2}(\epsilon x)^{-a}x^{a-1}\,dx + k_0'$$

$$= \frac{k}{\epsilon^a}\int_{1/\epsilon}^{\infty} x^{-(a+3)/2}\,dx + k_0' = k'\epsilon^{(a+1)/2-a} + k_0' = k'\mathfrak{T}^{(a-1)^2/2(a+1)} + k_0'.$$

Thus

$$\sum_{\Lambda\in L}{}' |\delta_\epsilon(\Lambda)g_T(\Lambda)| \le k_3\mathfrak{T}^{2b+(a-1)/2+(a-1)^2/2(a+1)} = k_3\mathfrak{T}^{2b+a(a-1)/(a+1)},$$

$$\sum_{\Lambda\in B(\mathfrak{T})}^{\bullet} |h_T(\Lambda) - g_T(\Lambda)| \le k\,\mathfrak{T}^{2b+a-1}\epsilon$$

$$= k\,\mathfrak{T}^{2b+a-1-(a-1)/(a+1)} = k\mathfrak{T}^{2b+a(a-1)/(a+1)}$$

so $R(T)$ is $O(\mathfrak{T}^{2b+a(a-1)/(a+1)})$. □

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